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2004 J. Phys. A: Math. Gen. 37 2115

(http://iopscience.iop.org/0305-4470/37/6/011)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **37** (2004) 2115–2120 PII: S0305-4470(04)70621-6

Vertex–face correspondence of Boltzmann weights related to $sl(m|n)$

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Received 20 October 2003, in final form 12 December 2003
Published 28 January 2004
Online at stacks.iop.org/JPhysA/37/2115 (DOI: 10.1088/0305-4470/37/6/011)

Abstract
In this work, we present a vertex–face correspondence between an elliptic R-operator and Boltzmann weights related to the Lie superalgebra $sl(m|n)$.

PACS numbers: 02.20.Uw, 02.30.Ik, 05.50.+q
Mathematics Subject Classification: 82B23

Dedicated to Professor Tatsuo Suwa on the occasion of his sixtieth birthday.

1. Introduction

A key method in solving two-dimensional lattice models of the face type is Baxter's corner transfer matrix method [1], which requires that Boltzmann weights satisfy the star–triangle relation (STR) and the inversion relations. Much attention has thus been directed to finding Boltzmann weights with the properties. The vertex–face correspondence is a major tool in constructing such Boltzmann weights.

Andrews *et al* [2] constructed the Boltzmann weights associated with the R-matrix of the eight-vertex model through a vertex–face correspondence [3]. By extending the work above, Jimbo *et al* [4–6] presented the Boltzmann weights related to the affine Lie algebra $A^{(1)}_{N-1}$ (see also [7]). In order to show the STR, they used a vertex–face correspondence whose vertex counterpart is Belavin's R-matrix [8]. An elliptic R-operator [9–12], a generalization [13] of Belavin's R-matrix, has a vertex–face correspondence [14] (cf [15]) which reproduces the vertex–face correspondence above. In 1991, Okado constructed the Boltzmann weights [16] related to the Lie superalgebra $sl(m|n)$, a generalization of the Boltzmann weights defined by Jimbo *et al*. They satisfy the STR and the inversion relations.

Until now, no work has focused on a vertex–face correspondence of the Boltzmann weights related to $sl(m|n)$.

In this paper, we investigate and present a vertex–face correspondence between the elliptic R-operator and the Boltzmann weights related to $sl(m|n)$, which is a generalization of the work [14].

Let us now explain how this paper is organized. In section 2, we survey the Boltzmann weights related to the Lie superalgebra $sl(m|n)$ and the elliptic R-operator. We note that the domain of the elliptic R-operator in this work is some simple algebraic extension field of $\mathcal{M}_2$, the field of functions meromorphic on $\mathbb{C}^2$. Section 3 describes the vertex–face correspondence between the elliptic R-operator and the Boltzmann weights related to $sl(m|n)$.

2. Elliptic *R*-operator and Boltzmann weights related to $sl(m|n)$

In this section we present a review of the elliptic R-operator and the Boltzmann weights related to the Lie superalgebra $sl(m|n)$.

Fix a complex number τ whose imaginary part is positive ($\mathrm{Im}\,\tau > 0$) and we choose a complex number η such that $\eta, 2\eta \notin \mathbb{Z} + \mathbb{Z}\tau$. We denote by $\theta(x)$ the elliptic theta function

$$\theta(x) = \sum_{k\in\mathbb{Z}} \exp\left[\pi\sqrt{-1}\left(k+\frac{1}{2}\right)^2 \tau + 2\pi\sqrt{-1}\left(k+\frac{1}{2}\right)\left(x+\frac{1}{2}\right)\right].$$

Let m and n be nonnegative integers such that $m + n > 0$ and let $\mathcal{A} = \{e_\mu | \mu = 1, 2, \ldots, m+n\}$ be a basis of the vector space $\mathbb{C}^{m+n}$. Define a $\mathbb{C}$-bilinear form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{m+n} \times \mathbb{C}^{m+n}$ by $\langle e_\mu, e_\nu\rangle = s_\mu \delta_{\mu\nu}$, where $\delta_{\mu\nu}$ is the Kronecker delta symbol and

$$s_\mu = \begin{cases} 1 & \text{for} \quad \mu = 1, \ldots, m \\ -1 & \text{for} \quad \mu = m+1, \ldots, m+n. \end{cases}$$

Let I_μ ($\mu = 1, 2, \ldots, m+n$) be a complex number. For $a, b, c, d \in \mathbb{C}^{m+n}$ and $u \in \mathbb{C}$, we indicate by $W\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\middle| u\right)$ the Boltzmann weights related to the Lie superalgebra $sl(m|n)$ constructed by Okado [16]:

$$W\left(\begin{matrix} a & c \\ b & d \end{matrix}\middle| u\right) = 0 \qquad \text{unless} \quad b-a, c-a, d-b, d-c \in \mathcal{A}$$

$$W\left(\begin{matrix} a & a+e_\mu \\ a+e_\mu & a+2e_\mu \end{matrix}\middle| u\right) = \frac{\theta(\eta + s_\mu u)}{\theta(\eta)}$$

$$W\left(\begin{matrix} a & a+e_\mu \\ a+e_\mu & a+e_\mu+e_\nu \end{matrix}\middle| u\right) = \frac{\theta(\eta a_{\mu\nu} + I_{\mu\nu} - u)}{\theta(\eta a_{\mu\nu} + I_{\mu\nu})} \qquad (\mu \neq \nu)$$

$$W\left(\begin{matrix} a & a+e_\nu \\ a+e_\mu & a+e_\mu+e_\nu \end{matrix}\middle| u\right) = \frac{\theta(u)}{\theta(\eta)}\left(\frac{\theta(\eta a_{\mu\nu} + I_{\mu\nu} + \eta)\theta(\eta a_{\mu\nu} + I_{\mu\nu} - \eta)}{\theta(\eta a_{\mu\nu} + I_{\mu\nu})^2}\right)^{1/2} \quad (\mu \neq \nu).$$

Here $a_\mu = \langle a, e_\mu\rangle$, $a_{\mu\nu} = a_\mu - a_\nu$ and $I_{\mu\nu} = I_\mu - I_\nu$. They satisfy the STR:

$$\sum_{g\in\mathbb{C}^{m+n}} W\left(\begin{matrix} a & g \\ b & c \end{matrix}\middle| u\right) W\left(\begin{matrix} g & e \\ c & d \end{matrix}\middle| u+v\right) W\left(\begin{matrix} a & f \\ g & e \end{matrix}\middle| v\right)$$
$$= \sum_{g'\in\mathbb{C}^{m+n}} W\left(\begin{matrix} b & g' \\ c & d \end{matrix}\middle| v\right) W\left(\begin{matrix} a & f \\ b & g' \end{matrix}\middle| u+v\right) W\left(\begin{matrix} f & e \\ g' & d \end{matrix}\middle| u\right). \tag{1}$$

For a positive integer k, let $\mathcal{M}_k$ be the field of functions meromorphic on $\mathbb{C}^k$ and let $\overline{\mathcal{M}_k}$ be its algebraic closure. Denote by $h \in \overline{\mathcal{M}_2}$ a root of the following polynomial g in $\mathcal{M}_2[X]$:

$$g(X) = X^2 - \frac{\theta(x-y+\eta)\theta(x-y-\eta)}{\theta(x-y)^2}.$$

Proposition 1. *The minimal polynomial of h on the field $\mathcal{M}_2$ is g.*

Proof. For the proof, it is sufficient to show that $h \notin \mathcal{M}_2$. The proof is by contradiction. Assume the assertion was false. Then $h \in \mathcal{M}_2$, and the function $h(x, y)\theta(x-y)$ is holomorphic on $\mathbb{C}^2$ because

$$(h(x, y)\theta(x - y))^2 = \theta(x - y + \eta)\theta(x - y - \eta). \tag{2}$$

We indicate by $f(x, y)$ the holomorphic function $h(x, y)\theta(x - y)$ on $\mathbb{C}^2$. By equation (2),

$$f(x, 0)^2 = \theta(x + \eta)\theta(x - \eta)$$

and the right-hand side of the above equation consequently has a zero of the second order at the point $x = \eta$. This implies $2\eta \in \mathbb{Z} + \mathbb{Z}\tau$, which is a contradiction. □

Let $\mathcal{M}_2(h)$ be the simple algebraic extension field of $\mathcal{M}_2$ by h. Define an operator σ on $\mathcal{M}_2$ by

$$\sigma(f)(x, y) = f(y, x)$$

for $f \in \mathcal{M}_2$. The isomorphism σ on $\mathcal{M}_2$ is extended to an isomorphism on $\mathcal{M}_2(h)$, which carries h into h because the polynomial $g^\sigma(X) = X^2 - \sigma(\theta(x-y+\eta)\theta(x-y-\eta)/\theta(x-y)^2)$ $(= g(X))$ has a root h in $\mathcal{M}_2(h)$. Let us also denote by σ this isomorphism on $\mathcal{M}_2(h)$ since the other extension of σ on $\mathcal{M}_2(h)$ carries h into $-h$.

For $u \in \mathbb{C}$, define the elliptic R-operator $R(u)$ on $\mathcal{M}_2(h)$ [9–12] by

$$R(u)(f) = (\theta(u)/\theta(\eta))hf + B_u\sigma(f)$$

for $f \in \mathcal{M}_2(h)$, where $B_u(x, y) = \theta(x - y - u)/\theta(x - y) \in \mathcal{M}_2$.

Remark. The elliptic R-operator $R(u)$ satisfies the Yang–Baxter equation (YBE).

We indicate by $h_{ij} \in \overline{\mathcal{M}_3}$ $((i, j) = (1, 2), (1, 3), (2, 3))$ roots of the following polynomials g_{ij} in $\mathcal{M}_3[X]$ respectively:

$$g_{ij}(X) = X^2 - \frac{\theta(x_i - x_j + \eta)\theta(x_i - x_j - \eta)}{\theta(x_i - x_j)^2}.$$

The elements h_{ij} $((i, j) = (1, 2), (1, 3), (2, 3))$ satisfy the following lemma.

Lemma 2.

(1) $h_{ij} \notin \mathcal{M}_3$ *for* $(i, j) = (1, 2), (1, 3), (2, 3)$.
(2) $h_{13}, h_{23} \notin \mathcal{M}_3(h_{12})$ *and* $h_{23} \notin \mathcal{M}_3(h_{13})$.
(3) $h_{23} \notin \mathcal{M}_3(h_{12}, h_{13})$.

Here $\mathcal{M}_3(h_{12}, h_{13})$ *is the algebraic extension field of* $\mathcal{M}_2$ *by* h_{12} *and* h_{13}.

Let σ_{ij} $((i, j) = (1, 2), (1, 3), (2, 3))$ be an operator on $\mathcal{M}_3$ such that

$$\sigma_{ij}(f)(x_1, x_2, x_3) = f\left(x_{\bar{\sigma}_{ij}(1)}, x_{\bar{\sigma}_{ij}(2)}, x_{\bar{\sigma}_{ij}(3)}\right),$$

where

$$\bar{\sigma}_{ij}(k) = \begin{cases} j & \text{for } k = i \\ i & \text{for } k = j \\ k & \text{for } k \neq i, j. \end{cases}$$

The operator σ_{ij} $((i, j) = (1, 2), (1, 3), (2, 3))$ is an isomorphism on $\mathcal{M}_3$.

Lemma 3. *Let $\mathcal{M}_3(h_{12}, h_{13}, h_{23})$ be the algebraic extension field of $\mathcal{M}_3$ by h_{12}, h_{13} and h_{23}. For $(i, j) = (1, 2), (1, 3), (2, 3)$, the operator σ_{ij} is extended to an isomorphism on $\mathcal{M}_3(h_{12}, h_{13}, h_{23})$ which carries h_{kl} into $h_{\bar{\sigma}_{ij}(k)\bar{\sigma}_{ij}(l)}$. Here $h_{21} = h_{12}, h_{31} = h_{13}$ and $h_{32} = h_{23}$.*

From now on, we denote by σ_{ij} the extension of σ_{ij} in lemma 3. Define operators $R_{ij}(u)$ $(u \in \mathbb{C}, (i, j) = (1, 2), (1, 3), (2, 3))$ on $\mathcal{M}_3(h_{12}, h_{13}, h_{23})$ by

$$R_{ij}(u)(f) = (\theta(u)/\theta(\eta))h_{ij} f + (B_u)_{ij}\sigma_{ij}(f)$$

where $(B_u)_{ij}(x_1, x_2, x_3) = \theta(x_i - x_j - u)/\theta(x_i - x_j) \in \mathcal{M}_3$. These operators satisfy the YBE

$$R_{12}(u)R_{13}(u + v)R_{23}(v) = R_{23}(v)R_{13}(u + v)R_{12}(u) \qquad (u, v \in \mathbb{C})$$

because the functions $(B_u)_{ij}$ $((i, j) = (1, 2), (1, 3), (2, 3))$ satisfy the following functional equations:

$$\theta(u + v)(B_u)_{12}(B_v)_{13} = \theta(v)(B_{u+v})_{13}(B_v)_{32} + \theta(u)(B_v)_{23}(B_{u+v})_{12} \tag{3}$$

$$\frac{\theta(u)\theta(v)}{\theta(\eta)^2}(B_{u+v})_{13}\left(h_{12}^2 - h_{23}^2\right) = (B_v)_{23}(B_{u+v})_{12}(B_u)_{23} - (B_v)_{12}(B_{u+v})_{23}(B_u)_{12} \tag{4}$$

where $(B_v)_{32}(x_1, x_2, x_3) = \theta(x_3 - x_2 - v)/\theta(x_3 - x_2)$.

3. Vertex–face correspondence

In this section we establish a vertex–face correspondence between the elliptic R-operator and the Boltzmann weights related to $sl(m|n)$.

Let $\mathcal{M}_2'$ be a set of functions $f \in \mathcal{M}_2$ which satisfy that there exist two functions p_f, q_f holomorphic on $\mathbb{C}^2$ $(q_f \not\equiv 0)$ such that $f = p_f/q_f$, $q_f^{(\mu,\nu)} \not\equiv 0$, $\sigma(q_f)^{(\mu,\nu)} \not\equiv 0$ for all $\mu, \nu = 1, 2, \ldots, m + n$. Here, for a function f holomorphic on $\mathbb{C}^2$, we define the function $f^{(\mu,\nu)}$ holomorphic on $\mathbb{C}^{m+n}$ by

$$f^{(\mu,\nu)}(x_1, x_2, \ldots, x_{m+n}) = f(\eta x_\mu + I_\mu, \eta x_\nu + I_\nu + \eta s_\mu \delta_{\mu\nu}).$$

Remark. Every element $f \in \mathcal{M}_2$ is a quotient $f = p_f/q_f$ of two functions p_f, q_f holomorphic on $\mathbb{C}^2$ $(q_f \not\equiv 0)$ because of the Poincaré theorem [17].

Let $\mathcal{L}$ be a set of functions $f \in \mathcal{M}_2(h)$ such that $f = f_1 + f_2 h$ for $f_1, f_2 \in \mathcal{M}_2'$. The proof of the following lemma is straightforward and we omit it.

Lemma 4.

(1) $\mathcal{M}_2'$ is a subring of $\mathcal{M}_2$.
(2) $h^2 \in \mathcal{M}_2'$.
(3) $\mathcal{L}$ is a subring of $\mathcal{M}_2(h)$.
(4) $\sigma(\mathcal{M}_2') \subset \mathcal{M}_2'$ and $\sigma(\mathcal{L}) \subset \mathcal{L}$.
(5) $R(u)(\mathcal{L}) \subset \mathcal{L}$ for all $u \in \mathbb{C}$.

Let $h_{(\mu,\nu)} \in \overline{\mathcal{M}_{m+n}}$ $(1 \leqslant \mu < \nu \leqslant m + n)$ be a root of the following polynomial in $\mathcal{M}_{m+n}[X]$:

$$X^2 - \frac{\theta(\eta x_{\mu\nu} + I_{\mu\nu} + \eta)\theta(\eta x_{\mu\nu} + I_{\mu\nu} - \eta)}{\theta(\eta x_{\mu\nu} + I_{\mu\nu})^2}.$$

Here $x_{\mu\nu} = x_\mu - x_\nu$. We indicate by $h_{(\mu,\nu)}$ $(1 \leqslant \nu \leqslant \mu \leqslant m+n)$ the following element of $\overline{\mathcal{M}}_{m+n}$:

$$h_{(\mu,\nu)} = h_{(\nu,\mu)} \qquad \text{for} \quad 1 \leqslant \nu < \mu \leqslant m+n$$
$$h_{(\mu,\mu)} = 0 \qquad \text{for} \quad 1 \leqslant \mu \leqslant m+n.$$

For $f \in \mathcal{M}'_2$ and $\mu, \nu = 1, 2, \ldots, m+n$, define $f^{(\mu,\nu)} \in \mathcal{M}_{m+n}$ by

$$f^{(\mu,\nu)} = p_f^{(\mu,\nu)} / q_f^{(\mu,\nu)}.$$

We note that $f^{(\mu,\nu)}$ above is well defined and that $(h^2)^{(\mu,\nu)} = (h_{(\mu,\nu)})^2$.

Denote by $\phi^{(\mu,\nu)}$ $(\mu, \nu = 1, 2, \ldots, m+n)$ an operator from $\mathcal{L}$ to $\overline{\mathcal{M}}_{m+n}$ defined as follows:

$$\phi^{(\mu,\nu)}(f) = f_1^{(\mu,\nu)} + f_2^{(\mu,\nu)} h_{(\mu,\nu)}$$

for $f = f_1 + f_2 h \in \mathcal{L}$ $(f_1, f_2 \in \mathcal{M}'_2)$.

Lemma 5.

(1) $\phi^{(\mu,\nu)}$ *is a ring homomorphism.*
(2) If $\mu \neq \nu$, *then* $\phi^{(\nu,\mu)}\sigma|_{\mathcal{L}} = \phi^{(\mu,\nu)}$.

Proof. We only prove (2). If $\mu \neq \nu$, then we deduce the following, which immediately implies the desired result:

$$\sigma(f)^{(\nu,\mu)} = f^{(\mu,\nu)} \qquad \text{for} \quad f \in \mathcal{M}'_2;$$
$$h_{(\nu,\mu)} = h_{(\mu,\nu)}.$$

□

For $\mu, \nu, \kappa = 1, \ldots, m+n$ and $u \in \mathbb{C}$, let $W(\mu, \nu, \kappa|u)$ be an element of $\overline{\mathcal{M}}_{m+n}$ defined as follows:

$$W(\mu, \nu, \kappa|u) = 0 \qquad \text{unless} \qquad \kappa = \mu \text{ or } \nu;$$
$$W(\mu, \nu, \mu|u) = B_u^{(\mu,\nu)};$$
$$W(\mu, \nu, \nu|u) = (\theta(u)/\theta(\eta)) h_{(\mu,\nu)} \qquad \text{for} \quad \mu \neq \nu.$$

The form of $W(\mu, \nu, \kappa|u)$ is the same as the Boltzmann weight $W\left(\begin{smallmatrix} a & a+e_\kappa \\ a+e_\mu & a+e_\mu+e_\nu \end{smallmatrix}\middle| u\right)$.

Theorem 6 (Vertex–face correspondence)**.** *For* $\mu, \nu = 1, \ldots, m+n$ *and* $u \in \mathbb{C}$,

$$\phi^{(\mu,\nu)} R(u)\Big|_{\mathcal{L}} = \sum_{\kappa=1}^{m+n} W(\mu, \nu, \kappa|u) \phi^{(\kappa,\mu+\nu-\kappa)} \sigma \Big|_{\mathcal{L}}.$$

Proof. Let f be an element of $\mathcal{L}$. The straightforward computation shows

$$\phi^{(\mu,\nu)}(R(u)(f)) = (\theta(u)/\theta(\eta)) h_{(\mu,\nu)} \phi^{(\mu,\nu)}(f) + B_u \phi^{(\mu,\nu)} \sigma(f). \tag{5}$$

Due to lemma 5, we can show that the right-hand side of equation (5) turns out to be $\sum_{\kappa=1}^{m+n} W(\mu, \nu, \kappa|u) \phi^{(\kappa,\mu+\nu-\kappa)} \sigma(f)$, thereby completing the proof of the theorem. □

Remark. Equations (3) and (4) induce the STR (1) of the Boltzmann weights.

Acknowledgments

The author would like to express his gratitude to Professor Yuji Yamada for his useful suggestions. He would like to thank the referees for pointing out redundancies in the original manuscript. This research was partially supported by the Ministry of Education, Science, Sports and Culture, grant-in-aid for Young Scientists (B), 15740001, 2003.

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